

Utah State University

**DigitalCommons@USU**

---

Special Relativity

Notes

---

1-8-2018

## Special relativity, 2

David Peak

Utah State University, [david.peak@usu.edu](mailto:david.peak@usu.edu)

Follow this and additional works at: [https://digitalcommons.usu.edu/intermediate\\_modernphysics\\_special](https://digitalcommons.usu.edu/intermediate_modernphysics_special)



Part of the [Physics Commons](#)

---

### Recommended Citation

Peak, David, "Special relativity, 2" (2018). *Special Relativity*. Paper 2.  
[https://digitalcommons.usu.edu/intermediate\\_modernphysics\\_special/2](https://digitalcommons.usu.edu/intermediate_modernphysics_special/2)

This Course is brought to you for free and open access by the Notes at DigitalCommons@USU. It has been accepted for inclusion in Special Relativity by an authorized administrator of DigitalCommons@USU. For more information, please contact [digitalcommons@usu.edu](mailto:digitalcommons@usu.edu).



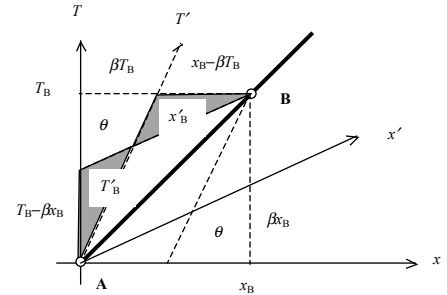
## Special relativity, 2

### Events not connected by light propagation

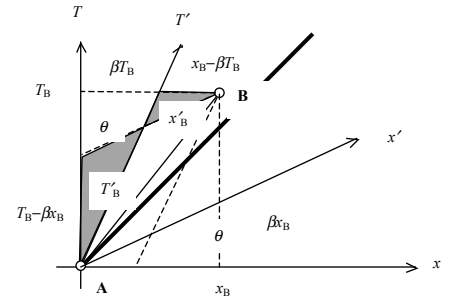
Previously, we considered two events (**A**, with s-t coordinates  $(x_A, y_A, z_A, T_A) = (x'_A, y'_A, z'_A, T'_A) = (0, 0, 0, 0)$  according to both O and O' [moving relative to O with constant velocity  $\beta$  along the mutual  $x, x'$ -axes], and **B**, with coordinates  $(x_B, 0, 0, T_B)$  and  $(x'_B, 0, 0, T'_B)$  connected by a light pulse. In order for both observers to conclude that the speed of the pulse was 1, we had to modify the Newtonian reckoning of time, namely, that (contrary to Newton)  $T'_B \neq T_B$ . In particular, we found that by modifying the standard Newtonian s-t diagram as shown to the right we could argue that

$$\frac{T'_B}{T_B - \beta x_B} = \frac{x'_B}{x_B - \beta T_B} \text{ and consequently derive the successful}$$

$$\text{velocity transformation rule } u'_{AB} = \frac{u_{AB} - \beta}{1 - \beta u_{AB}}.$$



A moment's reflection shows that there is actually nothing special about **B** being connected to **A** by a light pulse. In the s-t diagram to the right a signal between **A** and **B** travels slower than light. The labels are all essentially the same as in the previous s-t diagram. The similar triangles we used to derive the ratios above are still there, just with altered space and time coordinate values. The same velocity transformation rule as above can now be applied to signals not traveling at speed = 1. You might well wonder how come we didn't know this rule instead of the Newtonian rule  $u'_{AB} = u_{AB} - \beta$ . It's because for the circumstances of Newtonian physics velocities relative to  $c$  are typically puny.



**Example:** The New Horizons Pluto probe is the fastest large object yet produced on Earth. It left Earth traveling at a relative speed of about 16 km/s. Earth, on the other hand, is moving with a speed of about 370 km/s relative to the Cosmic Microwave Background (CMB). Thus, if the probe were heading in the same direction re the CMB as Earth, then according to Newton we would expect the probe velocity to be 386 km/s re the CMB. Let's suppose observer O is fixed to the CMB and O' is fixed to Earth. Then  $\beta = 370/c = 1.23 \times 10^{-3}$  and  $u' = 16/c = 0.05 \times 10^{-3}$  (where  $c = 300 \times 10^3$  km/s). What is  $u$ ? The roles of  $u$  and  $u'$  can be switched in either velocity transformation rule by replacing  $\beta$  by  $-\beta$ . Thus according to Newton  $u = 1.28 \times 10^{-3}$  while according to Einstein

$$u = 1.28 \times 10^{-3} / [1 + (1.28 \times 10^{-3})(1.23 \times 10^{-3})] = (0.999998)(1.28 \times 10^{-3}),$$

which, of course, among friends is the same thing. No wonder we didn't know how to properly add velocities before we started thinking about light.

Incidentally, we can make little things (like electrons) go much faster than the Pluto probe. For a circumstance where a particle traveling with speed  $= \frac{3}{4}$  relative to the laboratory emits another particle traveling relative to it, in the "forward" direction, with speed  $= \frac{3}{4}$ , the

speed of the second particle relative to the laboratory is not  $\frac{3}{4} + \frac{3}{4} = 1.5$ , but rather  $1.5/[1+(3/4)(3/4)] = 0.96$ .

If you examine the second s-t diagram you'll see that  $\frac{T'_B}{T_B - \beta x_B} = \frac{x'_B}{x_B - \beta T_B} > 1$ . Let's call this ratio  $\gamma$ . Thus, we can conclude that the rules of transformation between the  $x$  and  $T$  coordinates of O and O' are

$$x' = \gamma(x - \beta T) \quad (\text{as opposed to Newton's } x' = x - \beta T)$$

and

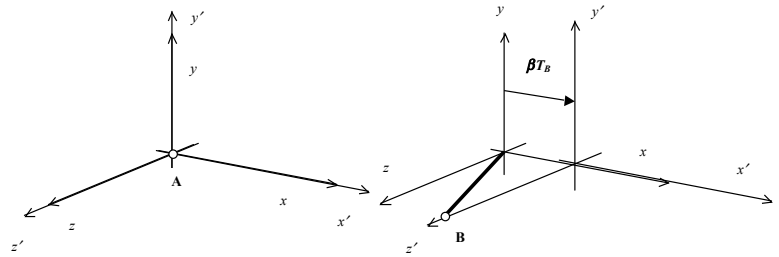
$$T' = \gamma(T - \beta x) \quad (\text{as opposed to Newton's } T' = T).$$

Notice, incidentally, the beautifully symmetric way in which the space and time coordinates (both measured as length) transform. This is the origin of the modern notion of "space-time" as opposed to Newton's more awkward "space AND time." If all we want to do is compute relative velocities along the x-axis we don't need to know what  $\gamma$  is, because  $\gamma$  cancels out when taking space to time ratios (velocities). But,  $\gamma$  has profound consequences for other kinematic and dynamic quantities of interest so evaluating it is essential. To do this, we make an appeal to symmetry: there is no difference between observers O and O'. According to O', O is traveling in the  $-x'$  direction and therefore O' should relate the coordinates of events in the two frames as  $x = \gamma(x' + \beta T')$  and  $T = \gamma(T' + \beta x')$ . Now, if we replace  $x'$  and  $T'$  in the latter equation by the expressions above we get  $x = \gamma^2(1 - \beta^2)x$ , which for arbitrary values of  $x$  implies

$$\gamma = \frac{1}{\sqrt{1 - \beta^2}}. \quad (1)$$

#### Events in other directions: Lorentz transformations

It's important to keep in mind what events look like in space pictures as well as in s-t diagrams. The figure to the right shows the *spatial* coordinate frames of O and O' at two times: when **A** occurs and later, when **B** occurs. The picture is drawn according to O. **A** is a flash of light



from the two overlapping spatial origins at  $T = T' = 0$ . Later, the flash is detected at a point on the  $z'$ -axis (event **B**). The distance the origin of O' has traveled between **A** and **B** along the  $x$ -axis is  $\beta T_B$ . Thus, O records **B** as taking place at  $x_B = \beta T_B, z_B, T_B$ , while O' records it at

$x'_B = 0, z'_B, T'_B$ . Using the time transformation rule above, we have  $T'_B = \gamma(1 - \beta^2)T_B$ . O' records the speed of signal propagation from **A** to **B** to be  $z'_B/T'_B = 1 = z'_B/[\gamma(1 - \beta^2)T_B]$ . O, on the other

hand, records the speed of signal propagation to be  $\sqrt{z_B^2 + \beta^2 T_B^2}/T_B = 1$ . The latter relation can be solved for  $T_B$ :  $T_B = z_B/\sqrt{1 - \beta^2}$ . If this expression is inserted into  $1 = z'_B/[\gamma(1 - \beta^2)T_B]$ , we

obtain  $\gamma = \frac{z'_B}{z_B} \frac{1}{\sqrt{1 - \beta^2}}$ . Because  $\gamma = \frac{1}{\sqrt{1 - \beta^2}}$ , we conclude that  $z'_B = z_B$ . The same result

would be obtained if we located **B** on the  $y'$ -axis. Thus, we conclude that **lengths perpendicular to the direction of relative motion don't change**.

We are now able to write down the correct and complete set of transformations for positions and times according to two observers in constant relative motion (whose origins coincide at  $T = T' = 0$ ):

$$\begin{aligned}T' &= \gamma(T - \beta x) \\x' &= \gamma(x - \beta T) \\y' &= y \\z' &= z\end{aligned}\tag{2}$$

Note that for two events separated by infinitesimal times and distances

$$\begin{aligned}dT' &= \gamma(dT - \beta dx) \\dx' &= \gamma(dx - \beta dT) \\dy' &= dy \\dz' &= dz\end{aligned}$$

The latter lead immediately to the velocity transformations:

$$\begin{aligned}u'_x &= \frac{u_x - \beta}{1 - \beta u_x} \\u'_y &= \frac{u_y}{\gamma(1 - \beta u_x)} \\u'_z &= \frac{u_z}{\gamma(1 - \beta u_x)}\end{aligned}\tag{3}$$

where  $u'_x = dx'/dT'$ ,  $u_x = dx/dT$ , and so forth. The coordinate transformations are called the *Lorentz transformations*. Together, the Lorentz [(2)] and velocity [(3)] transformations constitute Einstein's kinematic theory of *Special Relativity* ("special" because the relatively moving observers are both restricted to being inertial).